IMPROVING AND MAXIMAL INEQUALITIES IN DISCRETE HARMONIC ANALYSIS

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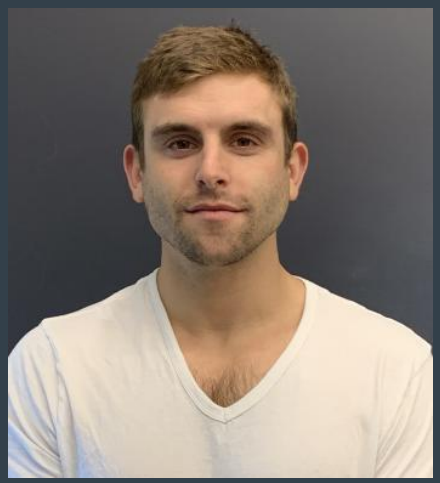
Dissertation Defense

June 7, 2023

Results discussed

- C.G., **Averaging with the divisor function: ℓ^p-improving and sparse bounds.** Rocky Mountain J. Math. 52(6): 2027-2039 (2022)
- C.G., M. T. Lacey, H. Mousavi, and Y. Rahimi **Improving and Maximal Inequalities for Primes in Progressions.** Banach J. Math. Anal. 16, 42 (2022)
- C.G., B. Krause, M. T. Lacey, H. Mousavi, and Y. Rahimi **Averages over the Gaussian Primes: Goldbach's Conjecture and Improving Estimates.** to be submitted to Journal of the London Mathematical Society (2023)

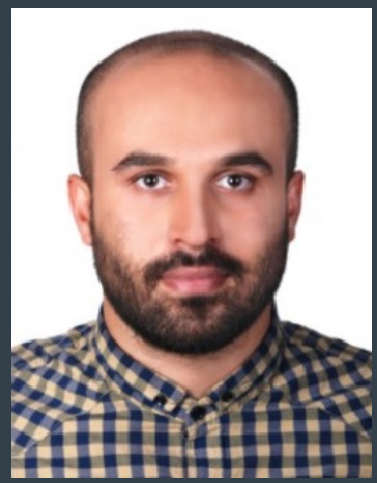
My Co-Authors



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History

Improving & Maximal Bounds

- Area of particular interest since the work of Littman and Strichartz in the early 70s.

  Operators, and families of operators can *improve* upon the p-norm by achieving bounds over *larger p-values*.

- *Hardy and Littlewood Maximal function* for studying cricket averages:

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy,$$

  where $B(x,r) = \{y \in \mathbb{R}^d : |x-y| < r\}$, and $|E|$ denotes the d-dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^d$.

Ergodic History

Birkhoff's Ergodic Theorem came in 1931.

Ergodic Theorems in the sense that we are interested in came to the spotlight with two landmark papers of Bourgain. For a measure-preserving transformation T:

- J. Bourgain, *On The Pointwise Ergodic Theorem on L_p for Arithmetic Sets (1988)*

  For $A_N f = \frac{1}{N} \sum_{1 \leq n \leq N} T^{(n)} f$ we have $\| \sup_N A_N f \|_{L_p} \leq C \| f \|_{L_p}$.

- J. Bourgain, *Pointwise ergodic theorems for arithmetic sets (1989)*

  For $A_N f = \frac{1}{N} \sum_{1 \leq n \leq N} T^{(Q(n))} f$ we have $\| \sup_N A_N f \|_{L_p} \leq C \| f \|_{L_p}$.

Discrete Setting

Maximal operators can be defined for discrete functions f defined on the integers. (two-sided sequences)

A very interesting question is to prove maximal inequalities for various families of averaging operators, both in the continuous and the discrete setting!

Q: How can we achieve that?

Tools & Techniques

Sparse families

Let $\mathcal{D} = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k$ be a dyadic lattice collection that partitions $\mathbb{R}^d$ into cubes of side-length 2^k and each $Q \in \mathcal{D}_k$ belongs to exactly one cube $R \in \mathcal{D}_{k+1}$.

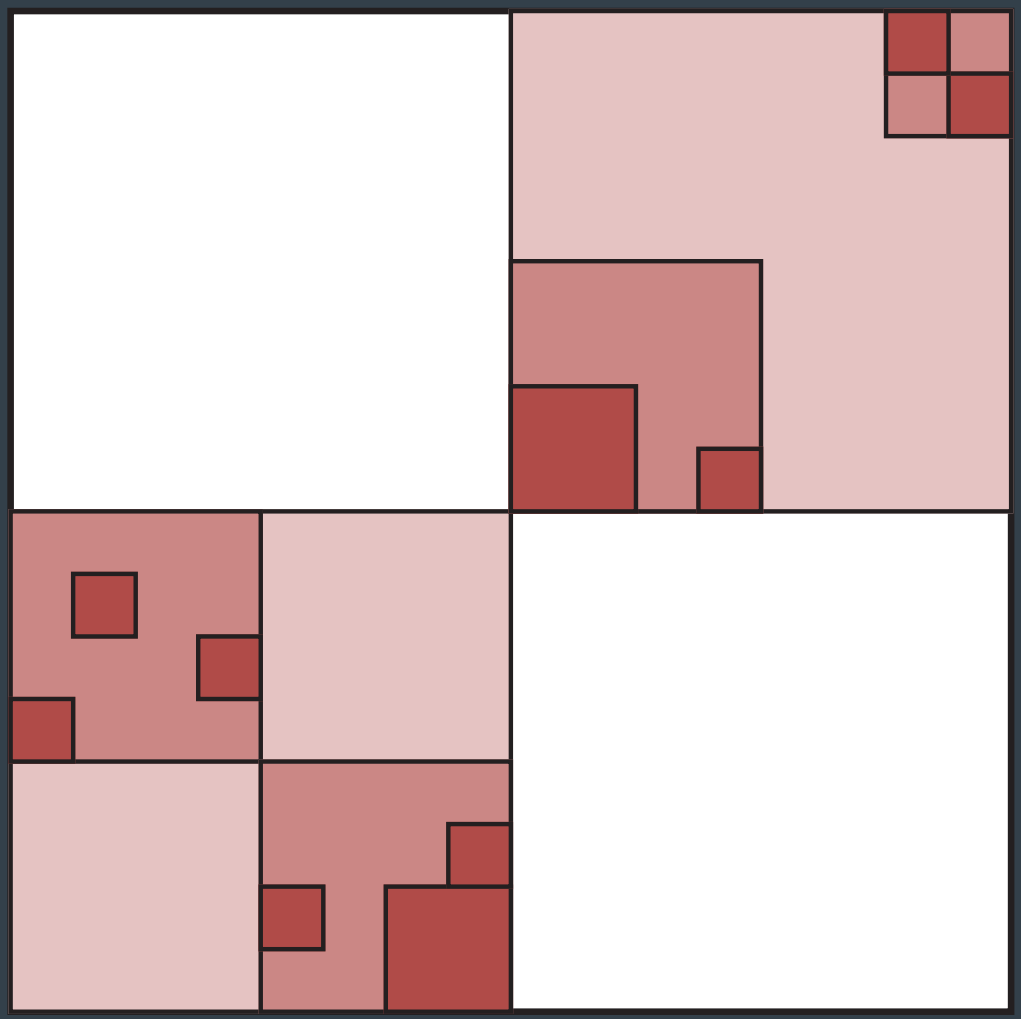
Definition

A family of cubes $\mathcal{S}$ in $\mathbb{R}^d$ is called 1/2-sparse if for any $Q \in \mathcal{S}$ we have

$$\left| \bigcup_{\substack{Q'\in\mathcal{S},\\ Q'\subsetneq Q}} Q' \right| \leq \frac{1}{2}|Q|.$$

Such a Q' is called a **child** of Q.

Visualization in $\mathbb{R}^2$



Parent

Q_0

Children

1st generation

2nd generation

3rd generation

Sparse Forms

Given a sparse collection $\mathcal{S}$, and for $1 \leq r, s < \infty$, an (r,s)-sparse form is

$$\Theta_{r,s}(f,g) = \sum_{Q \in \mathcal{S}} |Q| \langle f \rangle_{Q,r} \langle g \rangle_{Q,s}$$

where

$$\langle f \rangle_{Q,r} = \left(\frac{1}{|Q|} \int_Q |f|^r \right)^{1/r}.$$

When $r = 1$ it is often omitted.

Theorem

If an operator T is dominated by an (r,s) sparse form, then it is bounded $T : L^p(w) \to L^p(w)$, for all weights $w \in A^p$ and $p \in (r, s')$.

where A^p is the Muckenhaupt weight class.

The circle method

- The initial idea was attributed to G.H. Hardy (1877-1947) and S. Ramanujan (1887-1920) when studying the asymptotics of the partition function.

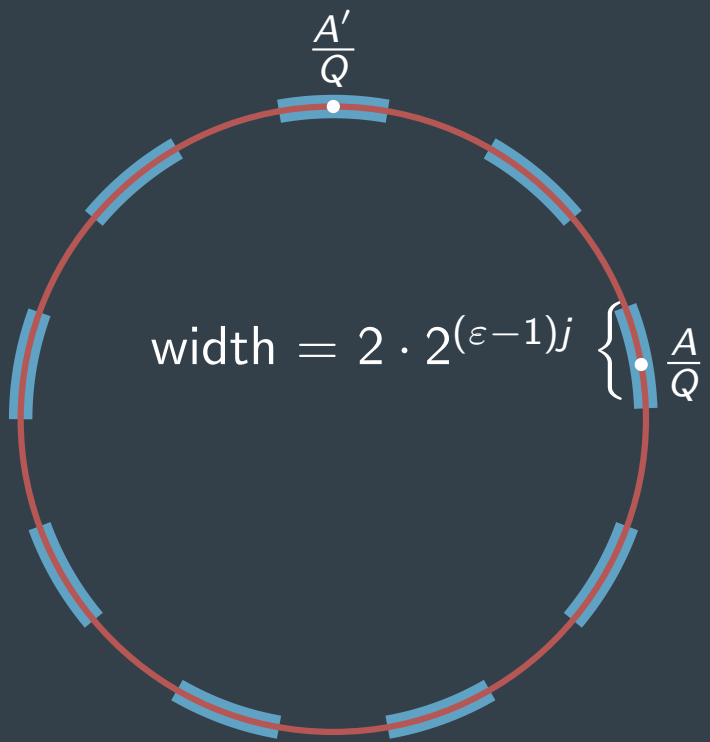
Main Idea

Heuristically, when looking at the 1-dimensional torus, we can split it into two pieces called **major** and **minor arcs**, and study their contributions separately. Major arcs contain 'almost' rationals whose denominators we can control. Everything else gets thrown in the minor arcs.

We can generalize this idea in higher dimensions!

Major and Minor arcs

Let $0 < \varepsilon \ll 1/4$, $s \geq 1$: $\quad \mathcal{R}_s := \left\{ \frac{A}{Q} \in \mathbb{T} \,:\, (A,Q) = 1,\ 2^{s-1} \leq Q < 2^s \right\}$



For $s < j\varepsilon$, the j-th major box at A/Q:

$$\mathfrak{M}_j(A/Q) := \left\{ a \in \mathbb{T} : |a - A/Q| \leq 2^{j(\varepsilon-1)} \right\}$$

The boxes are disjoint for ε small enough.

We define the j-th *major arc* as

$$\mathfrak{M}_j := \bigcup_{\frac{A}{Q} \in \mathcal{R}_s} \mathfrak{M}_j(A/Q)$$

Minor arcs are the complements of major arcs.

Fourier Truncation

Q: How do we utilize this in our results?

Bourgain's High Low Fourier Truncation Method



J. Bourgain, *"New global well-posedness results for non-linear Schrödinger equations"*, (1999).

Main Idea

Suppose we want to estimate an inner product.

$\rightarrow$ One way is to use Hölder: $|(f,g)| \leq \|f\|_p \|g\|_{p'}$.

We break $f = H + L$. That allows us to apply Hölder twice, using different dual indices in each part:

$$|(f,g)| = |(H+L,g)| \leq \|H\|_p \|g\|_{p'} + \|L\|_q \|g\|_{q'}.$$

Reductions

Suppose T is an averaging operator.

To obtain an improving estimate, by duality, it suffices for us to prove the following statement, restricted on indicators:

Lemma

Suppose E is an interval of length N and $p \in (1,2)$. If $f = \mathbb{1}_F$ supported on $F \subseteq E$ and $g = \mathbb{1}_G$ supported on $G \subseteq E$ then

$$\frac{1}{N}(Tf, g) \leq C_p \langle f \rangle_{E,p} \langle g \rangle_{E,p},$$

where C_p is independent of N.

Reductions

For the maximal estimates, the simplification is made possible thanks to a recursion argument introduced in



Robert Kesler, Micheal Lacey, and Dario Mena, *"Sparse bounds for the discrete spherical maximal functions"*, (2020)

Suppose E is an interval of length 2^{n_0}. Let $f = \mathbb{1}_F$ be supported on $F \subseteq E$ and $g = \mathbb{1}_G$ be supported on $G \subseteq E$. Now consider a choice of stopping time $\tau : E \to \{2^n : 1 \leq n \leq n_0\}$.

Definition

A stopping time τ is admissible if and only if for any interval $I \subseteq E$ such that $\langle f \rangle_{3I,1} > 100 \langle f \rangle_{2E,1}$, there holds $\inf\{\tau(x) : x \in I\} > |I|$.

Reductions

Therefore, for the maximal bound, it suffices to show:

Lemma

For all admissible stopping times, and for all $1 < p < 2$ we have that

$$(K_\tau f, g) \lesssim \left(\langle f \rangle_{2E,1} \langle g \rangle_{E,1}\right)^{1/p} |E|$$

Averages using the divisor function

Similar Problems

- R. Han, M. Lacey, and F. Yang, Averages along the square integers: ℓ^p improving and sparse inequalities (2019)
- R. Han, B. Krause, M. Lacey, and F. Yang, Averages along the primes: Improving and sparse bounds (2019)
- R. Han, V. Kovač, M. Lacey, J. Madrid, and F. Yang, Improving estimates for discrete polynomial averages (2019)
- S. Dasu, C. Demeter, and B. Langowski, Sharp ℓ^p-improving estimates for the discrete paraboloid (2020)

The Operator

Let $d(n)$ be the usual divisor function, and define their sum $D(N) = \sum_{n \leq N} d(n)$.

Now consider the normalized averaging operator

$$K_N f = \frac{1}{D(N)} \sum_{n \leq N} d(n) f(x+n)$$

and the associated maximal function

$$K^* f = \sup_N |K_N f|.$$

The Results

Theorem (C.G.)

For $p \in (1, \infty)$, there exists $C_p > 0$ such that for all positive integers N and functions f supported on an interval E of length N, there holds

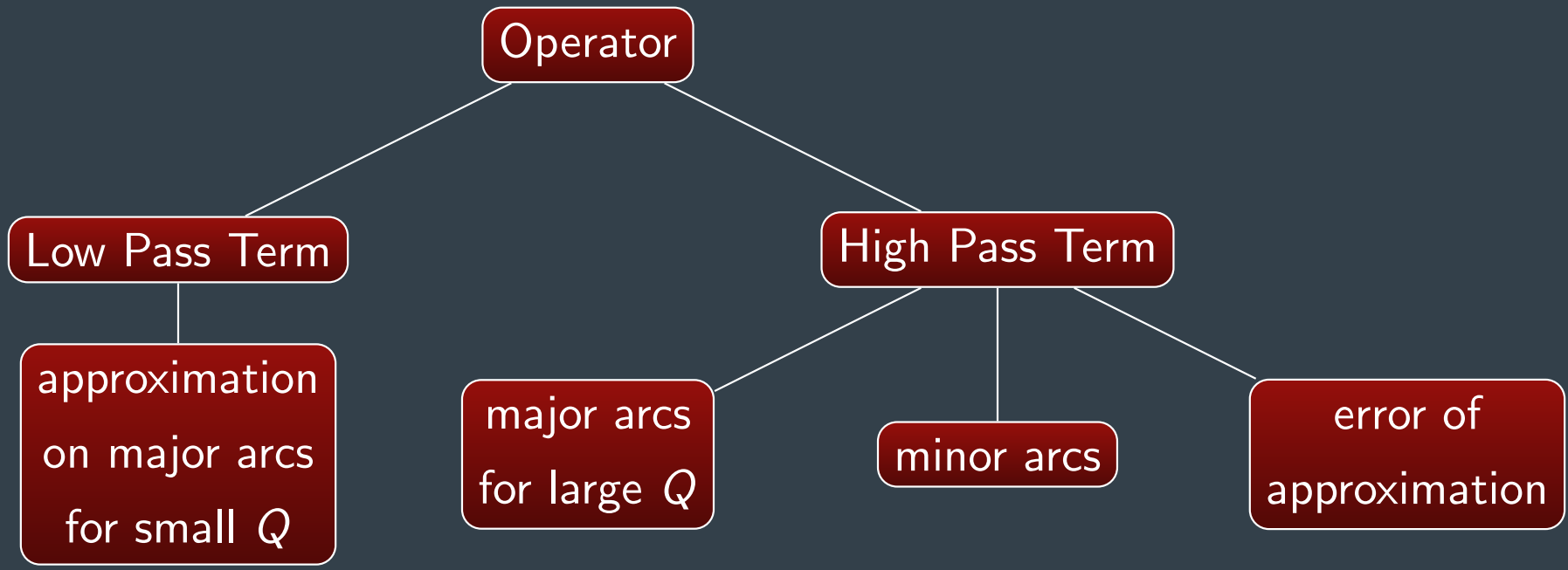
$$\langle K_N f \rangle_{E,p'} \leq C_p \langle f \rangle_{E,p}$$

Theorem (C.G.)

For $r, s \in (1, 2)$, the maximal operator K is of (r, s)-sparse type, that is there exists $C > 0$ such that for all compactly supported functions f and g, there exists a sparse collection $\mathcal{S}$ so that the $\ell^2(\mathbb{Z})$ inner product satisfies

$$|(K^* f, g)| \leq C \sum_{E \in \mathcal{S}} |I| \langle f \rangle_{2I,r} \langle g \rangle_{I,s}$$

Decomposition



Averages with primes in arithmetic progressions

Previous Results

- M. Wierdl, *Pointwise Ergodic Theorem Along the Prime Numbers*, (1988)
- M. Mirek and B.Trojan, *Cotlar's Ergodic Theorem Along the Prime Numbers*, (2013)
- B. Trojan, *Endpoint Estimates for the Maximal Function Over Prime Numbers*, (2019)
- M. Lacey, H. Mousavi, Y. Rahimi, *Endpoint ℓ^p Improving Estimates for Prime Averages*, (2021)

The operator

For $b, y \in \mathbb{N}$, $y \leq N$ and $F : \mathbb{Z} \to \mathbb{R}$, let

$$A_{N,y,b}(x) := \frac{\phi(y)}{N} \sum_{\substack{n<N \\ n\text{: prime} \\ n \equiv b \mod y}} \log(n) f(x-n)$$

where ϕ is the Euler totient function.

Of course we are only interested in the case where $(b, y) = 1$.

Results

Theorem (C.G., Lacey, Mousavi, Rahimi)

For $r \in (1,2)$, there exists $C_r > 0$, so that for all integers y, there is a $N_{r,y} > 0$ such that for all $N > N_{r,y}$ and compactly supported functions f,

$$\max_{(b,y)=1} \langle A_{N,y,b} f \rangle_{r'} \leq C_r \langle f \rangle_r .$$

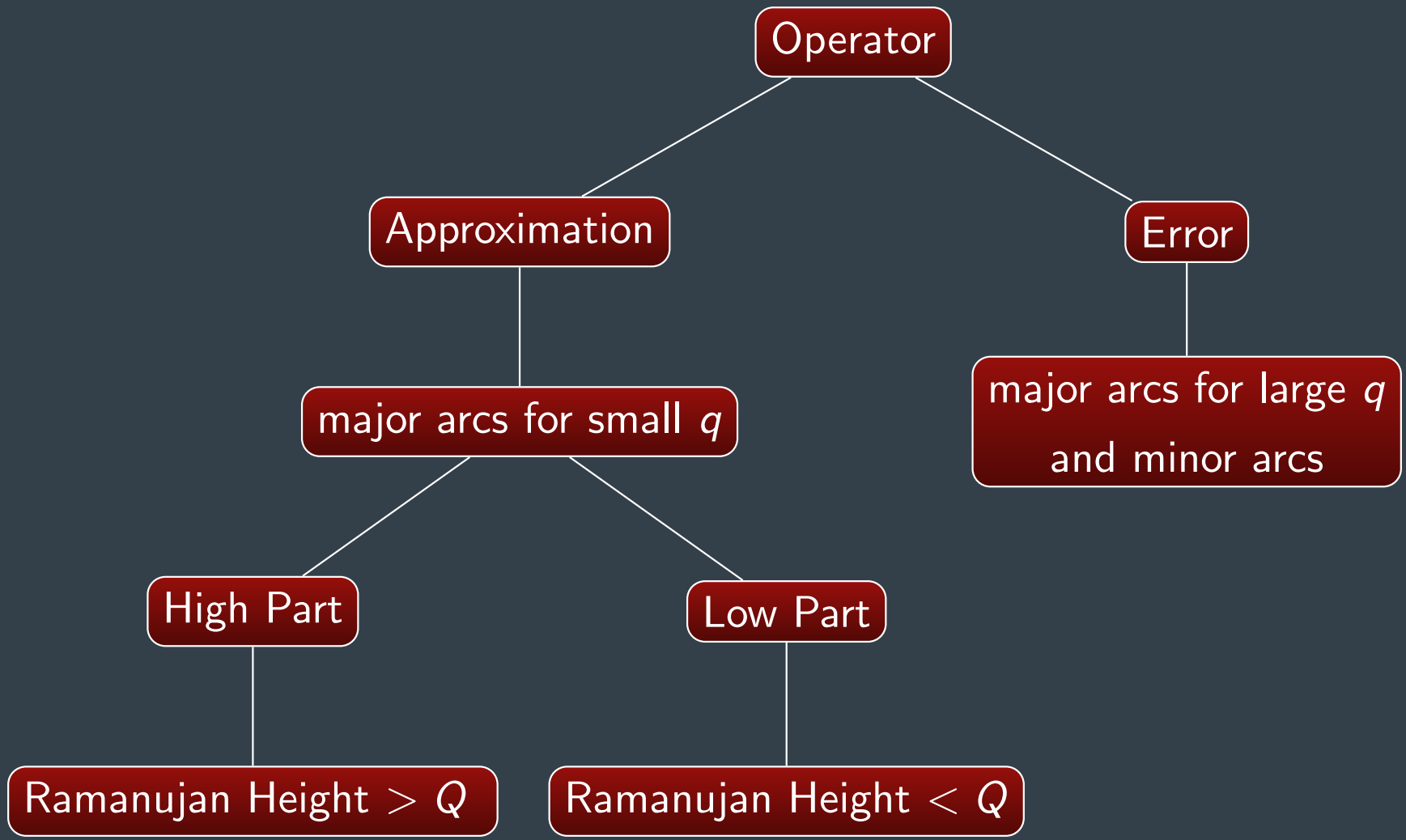
Above, we set $N_{r,y} = e^{C_{r,\delta} y^{\delta}}$, for any $\delta > 0$, and $C_{r,\delta}$ sufficiently large.

Theorem (C.G., Lacey, Mousavi, Rahimi)

For $1 < r < \infty$, there is a constant C_r so that for all integers y, there is an $N_{r,y} > 0$ so that uniformly in b and y,

$$\| \sup_{N > N_{r,y}} |A_{N,y,b} f| \|_{\ell^r} \leq C_r \|f\|_{\ell^r} .$$

Our Approach



Averages over the Gaussian Primes

The operator

Let $f : \mathbb{Z}[i] \to \mathbb{C}$ be an arithmetic function. For $x = a + bi \in \mathbb{Z}[i]$, define the norm $\mathcal{N}(z) := a^2 + b^2$, and let

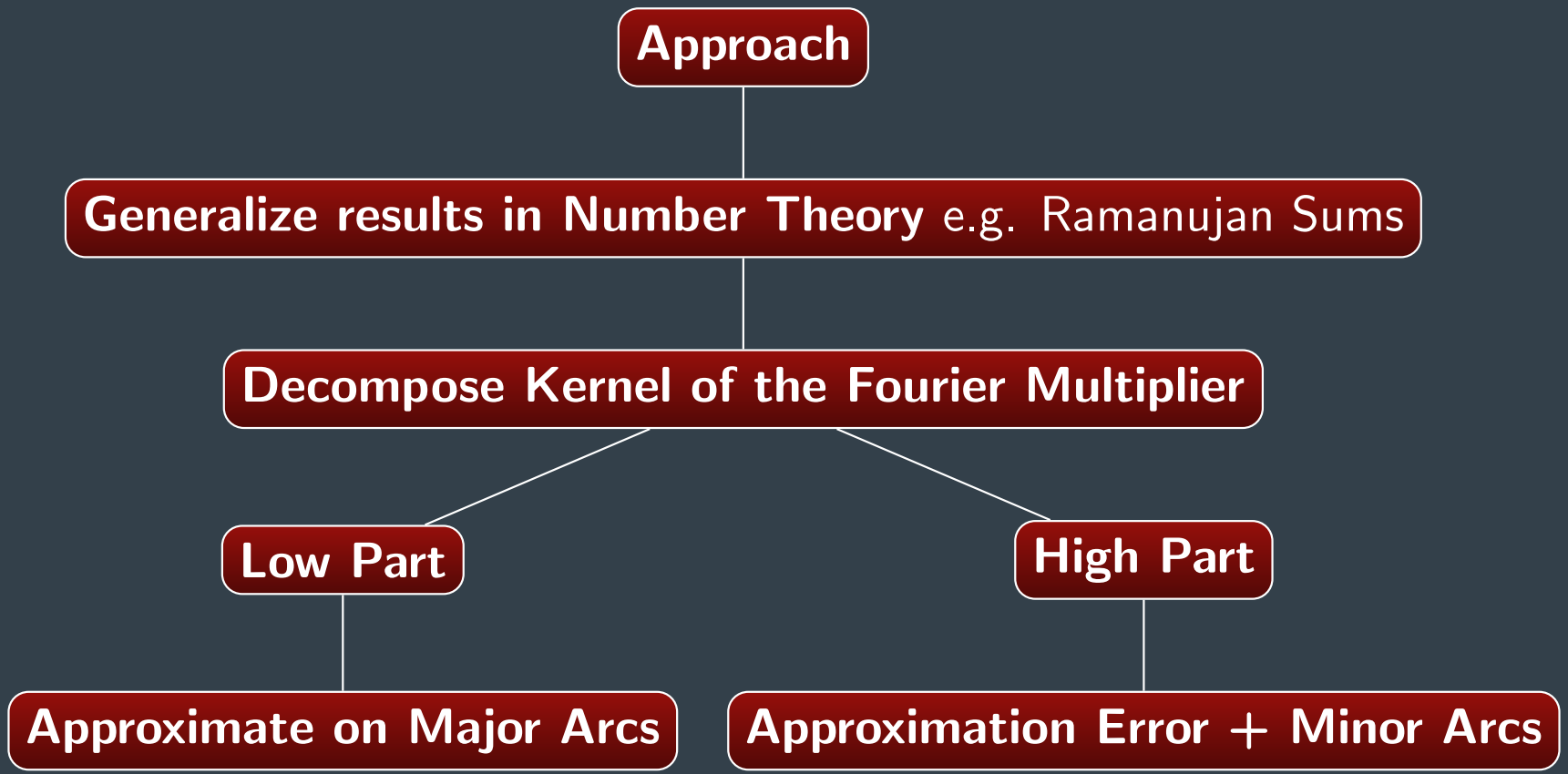
$$A_N f(x) = \frac{1}{N} \sum_{\substack{\mathcal{N}(n)<N \\ n\text{: prime}}} \log\left(\mathcal{N}(n)\right) f(x-n).$$

Theorem (C.G., Krause, Lacey, Mousavi, Rahimi)

For all $N \geq 1$, and $1 < p \leq 2$, the operator above satisfies the improving estimate

$$\langle A_N f, g \rangle \lesssim N^{1-2/p} \|f\|_p \|g\|_p.$$

Our Approach



Applications in Number Theory: Goldbach's Conjecture

Goldbach's Conjecture for real primes

Stated in the correspondence between C. Goldbach and L. Euler in the 1740's.

The Strong Goldbach Conjecture (Binary Goldbach)

Every even, positive integer $x \in \mathbb{Z}$ can be written as the sum of two prime numbers $p_1,\ p_2 \in \mathbb{Z}$.

The Weak Goldbach Conjecture (Ternary Goldbach)

Every positive integer $x \in \mathbb{Z}$ can be written as the sum of at most three prime numbers $p_1,\ p_2,\ p_3 \in \mathbb{Z}$.

The Complex Goldbach Conjecture

Question: How does one translate evenness in the Gaussian setting?

A Gaussian integer $x = a + ib$ is *even* if and only if $N(x) = a^2 + b^2$ is even.

This is equivalent to

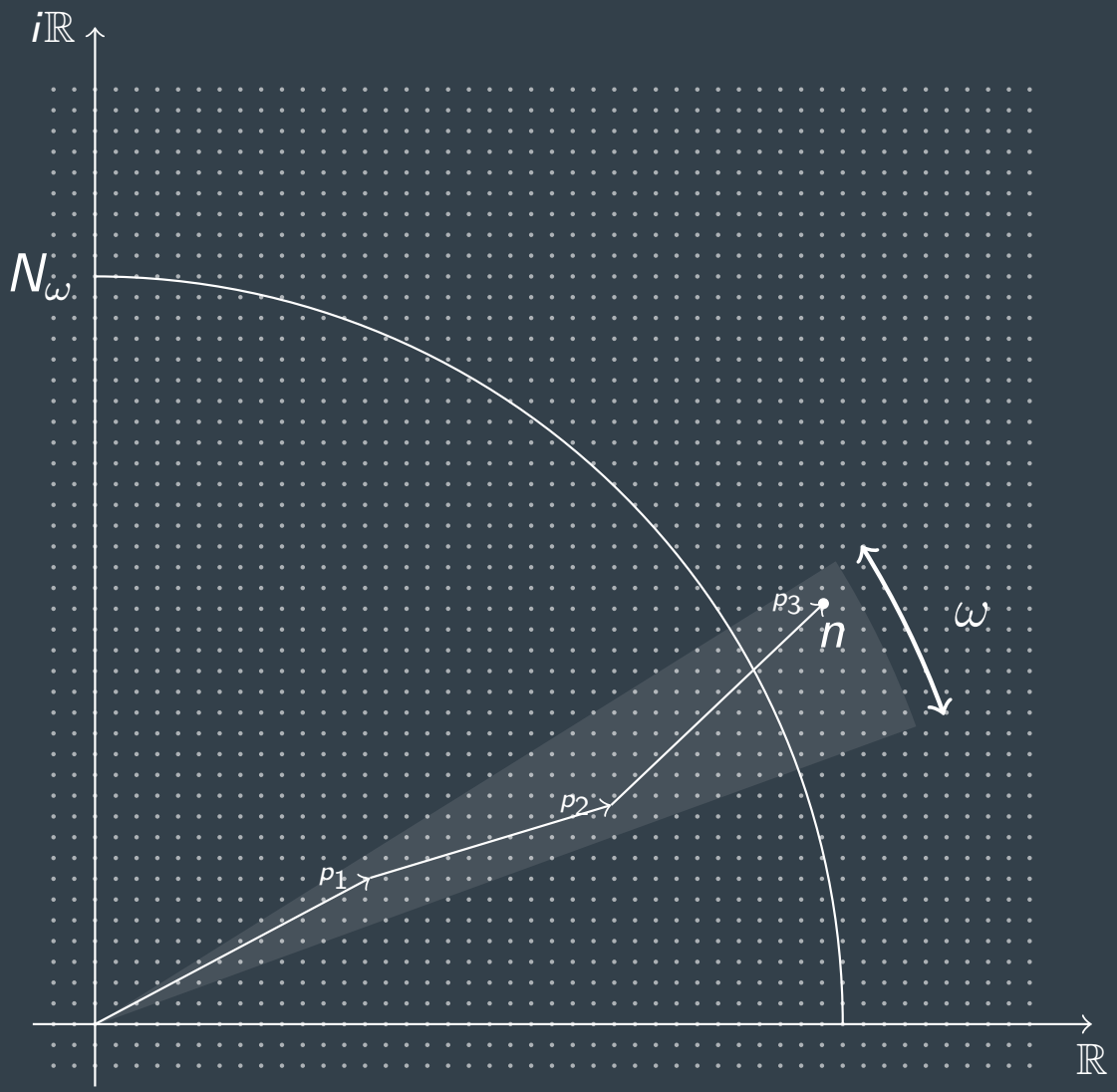
(i) The integer $a + b$ is even.

(ii) The Gaussian integer $1 + i$ divides x.

- Most natural extension is to restrict attention to the positive quadrant. A density version of the conjecture was addressed as early as Rademacher (1924) and Mitsui (1960), but the results assume the GRH.

Holben and Jordan raised the question in the context of sectors (1968).

Visualization of our result



- Pick an arbitrary sector ω.
- Find a radius N_ω large enough.
- Pick n odd, such that $\mathcal{N}(n) > N_\omega$.
- Write it as a sum of three primes $p_1,\ p_2,\ p_3$ in the sector, meaning $\arg(p_i) \in 2\pi\omega,\ i = 1, 2, 3$.

Formal Statement

Theorem (C.G., Krause, Lacey, Mousavi, Rahimi)

Fix an interval $\omega \subset \mathbb{T}$. There exists an $N_\omega > 0$, such that for all integers $N > N_\omega$, every odd integer n with $N(n) > N_{\omega,B}$, and $\arg(n) \in \omega$ is a sum of three Gaussian primes $n = p_1 + p_2 + p_3$, with $N(p_j) \leq N(n)$, and $\arg(p_j) \in \omega$ for $j = 1, 2, 3$.

- **To verify**: Use Hi-Lo estimates to construct upper bounds for the size of the exceptional sets.

$\longrightarrow$ Convolve the operator with itself twice!

$$\mathbb{1}_{\{x\text{: Prime}\}} * \mathbb{1}_{\{x\text{: Prime}\}} * \mathbb{1}_{\{x\text{: Prime}\}}(n)$$

Thank you!!!

Questions?